

Moufang Twin Trees of prime order

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We prove that the unipotent horocyclic group of a Moufang twin tree of prime order is nilpotent of class at most 2.

1 Introduction

The classification of spherical buildings asserts that each irreducible spherical building of rank at least 3 is of algebraic origin. By this we mean that it is the building of a classical group, or a semi-simple algebraic group, or some variation thereof. In the rank 2 case, this is no longer true; in particular, there are free constructions of generalized polygons. (Generalized polygons are precisely the spherical buildings of rank 2.) In order to characterize the generalized polygons of algebraic origin, Tits introduced the Moufang condition for spherical buildings in the 1970s [Tit77]. This condition is automatically satisfied for irreducible spherical buildings of rank at least 3. The Moufang polygons were classified in [TW02]. It follows from this classification that the Moufang condition characterizes indeed the generalized polygons of algebraic origin.

In the late 1980s Ronan and Tits introduced twin buildings, which were motivated by the theory of Kac-Moody groups. Twin buildings are generalizations of spherical buildings. For the latter there is a natural opposition relation on the set of its chambers due the existence of a unique longest element in the finite Weyl group. Many important results about spherical buildings (e.g. their classification in higher rank) rely on the presence of the opposition relation. For Kac-Moody groups over fields there is a natural notion of *opposite Borel groups*, even if its Weyl-group is infinite. The idea underlying the definition of twin buildings is to translate this algebraic fact into combinatorics. Roughly speaking the existence of an opposition relation for spherical buildings is axiomatized by the notion of a twinning between two buildings of the same (possibly non-spherical) type. It turns out that many important notions and concepts from the theory of spherical

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buildings have indeed natural analogues in the context of twin buildings. In particular, the Moufang condition makes sense for twin buildings. There is the natural question to which extent the “spherical” results can be generalized to the twin case. In this paper we contribute to this question in the context of twin trees which are precisely the non-spherical twin buildings of rank 2.

In view of the main result of [TW02] it is natural to ask, whether a classification of Moufang twin trees is feasible. Our main result can be seen as a major step towards a classification of Moufang twin trees of prime order (i.e. for regular Moufang trees of valency $p + 1$ for some prime p). This is of course a rather small subclass of all Moufang twin trees. As we shall explain below, however, a classification of all Moufang twin trees seems to be out of reach at the moment. In view of our result, there is some hope that a classification of the locally finite Moufang twin trees might be feasible. The latter are precisely the ones which are interesting for the theory of lattices in locally compact groups. Indeed, using a construction of Tits in [Tit89] and an important observation of Rémy in [Rém99] one knows that locally finite Moufang twin trees provide a large class of lattices in locally compact groups. The examples in this class are irreducible and non-uniform lattices in the full automorphism group of the product of two locally finite trees. Combining this with a result of Caprace and Rémy in [CR12] it turns out that a lot of them are simple as abstract groups. To our knowledge these are the only known examples of lattices with these properties. A classification of all locally finite Moufang trees would in particular provide a better understanding of these examples.

As already announced in the previous paragraph, we now provide more information about the classification problem for Moufang twin trees. We recall first that there is the natural question whether the Moufang condition characterizes the twin trees of algebraic origin, i.e., the examples provided by Kac-Moody groups and “their variations”. An important invariant of a Moufang twin tree is a subgroup of its automorphism group which is called its *unipotent horocyclic group*. In [Tit89] a general construction of Moufang twin trees is given which uses this invariant as an essential ingredient. In [RR06, Section 2] (see also [AR09, Example 67]) this construction was made “concrete” for certain parameters in order to construct “exotic” examples of Moufang twin trees with abelian unipotent horocyclic groups. In this way one gets classes of Moufang twin trees which one would not like to call of algebraic origin. Therefore the Moufang condition is not sufficient for characterizing the algebraic examples. Even worse, in [Tit96] it is shown that there are uncountably many non-isomorphic twin trees of valency 3. In view of the fact that for each value of n there is at most one Moufang n -gon of valency 3, one has to accept that the analogy of twin trees and generalized n -gons has its limitations.

On the other hand, at present it is not clear whether Moufang twin trees are “wild” or whether there is a powerful structure theory for them. This problem is discussed in [Tit89] and an abstract construction given therein provides a tool to obtain all Moufang twin trees. However, this has to be taken with a grain of salt because the procedure requires some group theoretical parameters. Hence, the construction given in [Tit89]

translates the classification problem for Moufang twin trees into the problem of classifying these parameters. The question whether these parameter sets can be classified is also discussed in [Tit89] and we briefly recall its outcome. First of all it turns out that a classification of all Moufang twin trees would provide a classification of all Moufang sets. Moufang sets have been studied intensively over the last 15 years and at present it seems that their classification is far beyond reach. As the finite Moufang sets are known (see e.g. [HKS72]) this difficult problem is not an obstacle if we restrict our attention to locally finite Moufang trees. However, there is still the problem of describing all possible commutation relations between the root groups in a Moufang twin tree for a given pair of Moufang sets. The main result of this paper provides a major step to solve this problem for Moufang twin trees of prime order. The commutation relations of a Moufang twin tree are in fact encoded in its unipotent horocyclic group mentioned before. The first step in our solution to the problem is to introduce \mathbb{Z} -systems, in order to axiomatize groups which are candidates for being the unipotent horocyclic group of a Moufang tree. We then prove Theorem 3.4, a purely group theoretical result whose statement requires some preparation. In order to give at least an idea about its implications for Moufang twin trees, we state the following consequence of it. As the precise definition of a Moufang twin tree won't be needed in the paper, we refer to [RT94] for an excellent introduction.

Theorem A. *The unipotent horocyclic group of a Moufang twin tree of prime order is nilpotent of class at most 2.*

As already mentioned, Theorem A is a consequence of our purely group theoretical Theorem 3.4. We indicate how Theorem A is deduced from Theorem 3.4 in Remark 3.5.

Let us finally point out the following two remarks on Theorem A:

- (i) As explained before, the theory of twin buildings was developed in order to provide the appropriate structures associated to Kac-Moody groups. Roughly speaking, the ingredients for defining such a group consist of a generalized Cartan matrix A and a field \mathbb{F} ; the resulting group is denoted by $G_A(\mathbb{F})$. If the Cartan matrix A is a 2×2 -matrix with non-positive determinant, then the twin building associated to $G_A(\mathbb{F})$ is a Moufang twin tree of order $|\mathbb{F}|$ whose automorphism group essentially coincides with the (adjoint version) of G_A . If A is of affine type (i.e. $\det(A) = 0$) then $G_A(\mathbb{F})$ can be realized as a matrix group over $\mathbb{F}(t)$. In fact, the examples given in Section 2 correspond to Kac-Moody groups of affine type. In most cases, however, $G_A(\mathbb{F})$ cannot be realized as a matrix group over a field (see [Cap09, Theorem 7.1]).
- (ii) We already mentioned that there are uncountably many pairwise non-isomorphic trivalent Moufang twin trees due to a construction of Tits given in [Tit96]. In view of our result above, one might hope that Tits' construction provides all trivalent Moufang twin trees which would give a classification of these objects. By modifying Tits' ideas we have constructed new examples which show that this is definitively

not the case. Nevertheless we are confident that a classification of Moufang twin trees of prime order is feasible. We intend to come back to this question in a subsequent paper.

Some conventions.

- We consider 0 to be a natural number, i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$.
- For a prime $p \in \mathbb{N}$, let $\mathbb{Z}_p := \{0, \dots, p-1\} \subset \mathbb{N}$ and $\mathbb{Z}_p^* := \{1, \dots, p-1\} \subset \mathbb{N}$. Moreover, let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ be the prime field of order p .
- For a group G , let $G^* := G \setminus \{1\}$.
- For $A, B, C \leq G$, set $[A, B, C] := [[A, B], C]$.
- For $U \subseteq G$, let $\langle U \rangle$ be the subgroup of G generated by U .

2 Moufang twin trees and RGD-systems

As explained in the introduction, the classification problem for Moufang twin trees can be translated into a purely group theoretical classification problem. The key notion on the group theoretic side is that of an RGD-system. We first outline what RGD-systems are, then review the interplay between Moufang twin trees and RGD-systems. This will provide the motivation for our main result and enable us to state it properly.

In [Tit92] RGD-systems have been introduced by Tits in order to investigate groups of Kac-Moody type and Moufang buildings. The abbreviation “RGD” stands for “root group data”. The axioms for an RGD-system are somewhat technical and we refer to [AB08] and to [CR09] for the general theory of RGD-systems.

Here we are only interested in RGD-systems of type \tilde{A}_1 , i.e. in RGD-systems whose type is the Coxeter system associated with the infinite dihedral group. The RGD-axioms given below are adapted to this special case in which they simplify considerably. This is because the root system Φ of type \tilde{A}_1 has the following concrete description.

Definition 2.1. For each $z \in \mathbb{Z}$ we put $\epsilon_z := 1$ if $z \leq 0$ and $\epsilon_z := -1$ if $z > 0$. We set $\Phi := \mathbb{Z} \times \{1, -1\}$, $\Phi^+ := \{(z, \epsilon_z) \mid z \in \mathbb{Z}\}$ and $\Phi^- := \Phi \setminus \Phi^+$. For $i = 0, 1$ we define $r_i \in \text{Sym}(\Phi)$ by $(z, \epsilon) \mapsto (2i - z, -\epsilon)$ and we put $\alpha_i := (i, \epsilon_i)$. Finally, for $\alpha = (z, \epsilon) \in \Phi$ we put $-\alpha := (z, -\epsilon)$.

Definition 2.2. An *RGD-system of type \tilde{A}_1* is a triple $\Pi = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ consisting of a group G , a subgroup H of G and a family $(U_\alpha)_{\alpha \in \Phi}$ of subgroups of G (the *root subgroups*) such that the following holds.

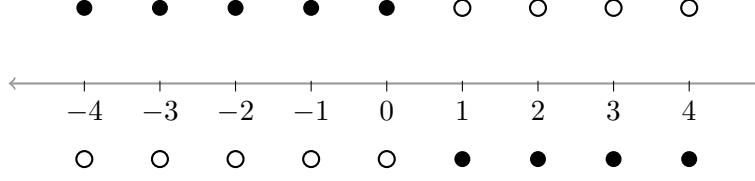


Figure 1: Root system of type \tilde{A}_1 ; black nodes are positive roots, white nodes negative roots.

(RGD1) For all $\alpha \in \Phi$ we have $|U_\alpha| > 1$.

(RGD2) For all $z < z' \in \mathbb{Z}$ and all $\epsilon \in \{1, -1\}$ we have

$$[U_{(z,\epsilon)}, U_{(z',\epsilon)}] \in \langle U_{(n,\epsilon)} \mid z < n < z' \rangle.$$

(RGD3) For $i = 0, 1$ there exists a function $m_i : U_{(i,1)}^* \rightarrow G$ such that for all $u \in U_{\alpha_i}^*$ and $\alpha \in \Phi$ we have

$$m_i(u) \in U_{-\alpha_i} u U_{-\alpha_i} \quad \text{and} \quad m_i(u) U_\alpha m_i(u)^{-1} = U_{r_i(\alpha)}.$$

Moreover, $m_i(u)^{-1} m_i(v) \in H$ for all $u, v \in U_{\alpha_i}^*$.

(RGD4) For $i = 0, 1$ the group $U_{-\alpha_i}$ is not contained in $\langle U_\alpha \mid \alpha \in \Phi^+ \rangle$.

(RGD5) The group G is generated by the family $(U_\alpha)_{\alpha \in \Phi}$ and the group H .

(RGD6) The group H normalizes U_α for each $\alpha \in \Phi$.

Remark 2.3. We refer to [AB08, Definition 7.82 and Subsection 8.6.1] for the definition of RGD-systems of arbitrary type. In the following discussion “RGD-system” shall always mean “RGD-system of type \tilde{A}_1 ”.

Example 2.4 (The standard example). Let \mathbb{F} be a field and set

$$G := \mathrm{SL}_2(\mathbb{F}[t, t^{-1}]) \leq \mathrm{SL}_2(\mathbb{F}(t)), \quad H := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid 0 \neq \lambda \in \mathbb{F} \right\} \leq G.$$

For each $z \in \mathbb{Z}$ we put

$$U_{(z,1)} := \left\{ \begin{pmatrix} 1 & \lambda t^z \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{F} \right\}, \quad U_{(z,-1)} := \left\{ \begin{pmatrix} 1 & 0 \\ \lambda t^{-z} & 1 \end{pmatrix} \mid \lambda \in \mathbb{F} \right\}.$$

We point out the following facts:

- (i) $\Pi = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ is an RGD-system.
- (ii) Let $U_{++} := \langle U_{(z,1)} \mid z \in \mathbb{Z} \rangle$. Then $U_{++} = \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \mid f \in \mathbb{F}[t, t^{-1}] \right\}$ and in particular $[U_{(z,1)}, U_{(z',1)}] = 1$ for all $z, z' \in \mathbb{Z}$.

(iii) $\langle U_{(z,1)}, U_{(z,-1)} \rangle$ is isomorphic to $\mathrm{SL}_2(\mathbb{F})$ for all $z \in \mathbb{Z}$.

Remark 2.5. The following aspect of the standard example is relevant in our context: Let ν be a place of $\mathbb{F}(t)$. Then $\mathrm{SL}_2(\mathbb{F}(t))$ acts on the Bruhat-Tits tree T_ν associated with ν . We consider the two rational places ∞ and 0 and set $T_+ := T_\infty$ and $T_- := T_0$. It is a fact that there is a twinning δ^* between T_+ and T_- such that $G = \mathrm{SL}_2(\mathbb{F}[t, t^{-1}])$ acts on the corresponding Moufang twin tree $T = (T_+, T_-, \delta^*)$ (see [RT94] for details). Moreover, the unipotent horocyclic group associated with T can be identified with the group U_{++} defined above.

The interplay between the RGD-system of $\mathrm{SL}_2(\mathbb{F}[t, t^{-1}])$ and the twin tree T is actually a special case of a general correspondence between RGD-systems and Moufang twin trees: It follows from [AB08, Proposition 8.22] that each Moufang twin tree T yields an RGD-system $\mathbf{\Pi}(T)$ in a canonical way. Conversely, for each RGD-system $\mathbf{\Pi}$, by [AB08, Theorem 8.81] there is a canonical associated twin tree $\mathbf{T}(\mathbf{\Pi})$. This correspondence is not one-to-one, but it can be made one-to-one by restricting to RGD-systems of “adjoint type”.

The following two facts about the correspondence between RGD-systems and Moufang twin trees are important in our context. Let $\mathbf{\Pi} = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ be an RGD-system and let $\mathbf{T}(\mathbf{\Pi})$ be the Moufang twin tree associated with $\mathbf{\Pi}$.

- (i) As a byproduct of the proof of [AB08, Theorem 8.81] one observes that the Moufang twin tree $\mathbf{T}(\mathbf{\Pi})$ is biregular of degree $(|U_{\alpha_0}| + 1, |U_{\alpha_1}| + 1)$. In analogy to the theory of projective planes, we say a tree is of *order* $q \in \mathbb{N}$ if it is a regular tree of degree $q + 1$.
- (ii) The group $U_{++} := \langle U_{(z,1)} \mid z \in \mathbb{Z} \rangle$ corresponds to the unipotent horocyclic group of $\mathbf{T}(\mathbf{\Pi})$.

Example 2.6 (The unitary example). Theorem A in the introduction asserts that the unipotent horocyclic group of a Moufang twin tree of order p is nilpotent of class at most 2. In the following we want to provide an example of an RGD-system $\mathbf{\Pi}$ which can be realized as a matrix group and such that the unipotent horocyclic group of \mathbf{T} is non-abelian. As this won't be used in the sequel, we omit the details.

Let \mathbb{F} be field with $\mathrm{char}(\mathbb{F}) \neq 2$. We define the following elements of $\mathrm{SL}_3(\mathbb{F}(t))$ for $z \in \mathbb{Z}$ and $\lambda \in \mathbb{F}$:

$$x_{2z}(\lambda) := \begin{pmatrix} 1 & -\lambda t^z & (-1)^{z+1} \frac{\lambda^2}{2} t^{2z} \\ & 1 & \lambda(-t)^z \\ & & 1 \end{pmatrix}, \quad x_{2z+1}(\lambda) := \begin{pmatrix} 1 & 0 & (-1)^z \lambda t^{2z+1} \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

$$h(\lambda) := \begin{pmatrix} \lambda & & \\ & 1 & \\ & & \lambda^{-1} \end{pmatrix}.$$

Moreover, we define the following subgroups:

$$\begin{aligned} U_{(2z+1,1)} &:= \{x_{2z+1}(\lambda) \mid \lambda \in \mathbb{F}\}, & U_{(2z,1)} &:= \{x_{2z}(\lambda) \mid \lambda \in \mathbb{F}\}, \\ U_{(2z+1,-1)} &:= U_{(2z+1,1)}^t, & U_{(2z,-1)} &:= U_{(2z,1)}^t, \\ H &:= \{h(\lambda) \mid \lambda \in \mathbb{F}^*\}. \end{aligned}$$

We set $G := \langle U_\alpha \mid \alpha \in \Phi \rangle$. The following can be verified by straightforward calculations.

- We have $H \leq G$.
- $\Pi = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ is an RGD-system.
- Each U_α is isomorphic to the additive group of \mathbb{F} .
- $U_{++} := \langle U_{z,1} \mid z \in \mathbb{Z} \rangle$ is non-abelian. Indeed, while the root groups $U_{2z+1,1}$ are central, we have for $z, z' \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{F}$ that

$$\begin{aligned} [x_{4z}(\lambda), x_{4z'+2}(\mu)] &= x_{2z+2z'+1}(2\lambda\mu), \\ [x_{4z+2}(\lambda), x_{4z'}(\mu)] &= x_{2z+2z'+1}(-2\lambda\mu), \\ [x_{4z}(\lambda), x_{4z'}(\mu)] &= [x_{4z+2}(\lambda), x_{4z'+2}(\mu)] = 1_G. \end{aligned}$$

3 The main result

As consequence of the discussion in the previous section, we conclude that the classification of Moufang twin trees of prime order p is equivalent to the classification of RGD-systems in which all U_α have order p . The Moufang sets of cardinality $p+1$ are classified. Thus, the main obstacle remaining in the classification of Moufang twin trees of prime order is the classification of the possible commutation relations. In order to make this more concrete, first consider the following basic observation about RGD-systems.

Lemma 3.1. *Let $\Pi = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ be an RGD-system. Let $X_n := U_{(n,1)}$ for each $n \in \mathbb{Z}$ and $X := \langle X_n \mid n \in \mathbb{Z} \rangle$. Then the following hold.*

- (i) *For all $n \leq m \in \mathbb{Z}$ the product map $X_n \times X_{n+1} \times \cdots \times X_m \rightarrow \langle X_i \mid n \leq i \leq m \rangle$ is a bijection.*
- (ii) *There exists $t \in \text{Aut}(X)$ such that $t(X_n) = X_{n+2}$ for all $n \in \mathbb{Z}$.*

Proof. Assertion (i) follows from Assertion (i) of Corollary 8.34 in [AB08]. Let $i = 0, 1$. Using the function m_i from (RGD3), we can construct $s_i \in G$ such that $U_\alpha^{s_i} = U_{r_i(\alpha)}$ for all $\alpha \in \Phi$. Then the mapping $t : X \rightarrow X, x \mapsto x^{s_0 s_1}$ has the required properties. \square

As we are dealing with Moufang twin trees of prime order, we have to consider RGD-systems in which all the U_α have order p for some prime number p . Let $\Pi = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ be such an RGD-system, and let X , $(x_n)_{n \in \mathbb{Z}}$ and t be as in the previous lemma. By choosing $1 \neq x_i \in U_{(i,1)}$ for $i = 0, 1$ and setting $x_{2n} := t^n(x_0)$ and $x_{2n+1} := t^n(x_1)$, we obtain a pair $(X, (x_n)_{n \in \mathbb{Z}})$ conforming to the following definition.

Definition 3.2. Let p be a prime. A \mathbb{Z} -system (of order p) is a pair $(X, (x_n)_{n \in \mathbb{Z}})$ consisting of a group X and a family $(x_n)_{n \in \mathbb{Z}}$ of elements in X such that the following conditions are satisfied.

(ZS1) $X = \langle x_n \mid n \in \mathbb{Z} \rangle$.

(ZS2) For all $n \leq m \in \mathbb{Z}$ the group $\langle x_k \mid n \leq k \leq m \rangle$ is of order p^{m-n+1} .

(ZS3) There exists an automorphism t of X such that $t(x_n) = x_{n+2}$ for all $n \in \mathbb{Z}$.

Example 3.3. Let p be a prime and let $\mathbb{F} := \mathbb{F}_p$.

- (i) Let everything be as in Example 2.4. For $n \in \mathbb{Z}$ let $u_n := \begin{pmatrix} 1 & t^n \\ 0 & 1 \end{pmatrix}$. Then $(U_{++}, (u_n)_{n \in \mathbb{Z}})$ is a \mathbb{Z} -system of order p . Indeed, the map

$$U_{++} \rightarrow U_{++}, \quad g \mapsto g^\sigma, \quad \text{where} \quad \sigma := \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

is an automorphism of U_{++} which maps u_n to u_{n+2} for all $n \in \mathbb{Z}$.

- (ii) Let everything be as in Example 2.6. For $n \in \mathbb{Z}$ let $u_n := x_n(1_{\mathbb{F}_p})$. Then $(U_{++}, (u_n)_{n \in \mathbb{Z}})$ is a \mathbb{Z} -system of order p . Indeed, the map

$$U_{++} \rightarrow U_{++}, \quad g \mapsto g^\sigma, \quad \text{where} \quad \sigma := \begin{pmatrix} t^{-1} & & \\ & 1 & \\ & & -t \end{pmatrix}$$

is an automorphism of U_{++} which maps u_n to u_{n+2} for all $n \in \mathbb{Z}$.

We already mentioned in the introduction that Tits gave a construction of uncountably many pairwise non-isomorphic trivalent twin trees. The idea behind his construction can be generalized to produce uncountably many non-isomorphic \mathbb{Z} -systems of order p for each prime p . It is conceivable that only very few of them can be realized as matrix groups. In a sense, Axiom (ZS3) requires an analogue of the conjugation by a diagonal matrix in the non-linear context.

We are now in the position to state our main result, which we prove in Section 9.

Theorem 3.4. *Let $(X, (x_n)_{n \in \mathbb{Z}})$ be a \mathbb{Z} -system of prime order. Then X is nilpotent of class at most 2.*

Remark 3.5 (Sketch of the proof of Theorem A). Let \mathbf{T} be a Moufang twin tree of order p and let $\Pi(T) = (G, (U_\alpha)_{\alpha \in \Phi}, H)$ be the RGD-system associated with T . As T is of order p , each U_α has order p . By Lemma 3.1 we therefore obtain a \mathbb{Z} -system $(X, (x_n)_{n \in \mathbb{Z}})$ of order p , and the unipotent horocyclic group of T coincides with X . Thus Theorem A is a consequence of Theorem 3.4

4 \mathbb{Z} -systems

For the rest of this paper, we assume that p is a prime and that $\Theta = (X, (x_n)_{n \in \mathbb{Z}})$ is a \mathbb{Z} -system of order p , together with an automorphism $t \in \text{Aut}(X)$ as in (ZS3), the *shift automorphism* of Θ . In the following lemma we collect some basic properties of \mathbb{Z} -systems.

Definition 4.1. For $n \leq m \in \mathbb{Z}$, we set

$$X_{n,m} := \langle x_k \mid n \leq k \leq m \rangle, \quad X_{-\infty,m} := \langle x_k \mid k \leq m \rangle, \quad X_{n,\infty} := \langle x_k \mid n \leq k \rangle.$$

Lemma 4.2. *The following statements are true.*

- (ZS4) *For each $n \in \mathbb{Z}$ we have $x_n^p = 1 \neq x_n$.*
- (ZS5) *For $n < m \in \mathbb{Z}$ we have $[x_n, x_m] \in X_{n+1,m-1}$.*
- (ZS6) *For each $x \in X^*$ there exist $n \leq m \in \mathbb{Z}$ and $e_n, \dots, e_m \in \mathbb{Z}_p$ such that $x = x_n^{e_n} \cdots x_m^{e_m}$, and both $e_n \neq 0$ and $e_m \neq 0$. Moreover, n, m, e_n, \dots, e_m are uniquely determined by x .*

Proof. (ZS4) is immediate from (ZS2) with $m = n$. Now recall that a subgroup of index p in a finite p -group is normal. Hence for any $n \leq m \in \mathbb{Z}$, we obtain the following normal series, where each group has index p in the preceding one:

$$X_{n,m} \triangleright X_{n+1,m} \triangleright \cdots \triangleright X_{m,m} \triangleright 1.$$

Thus x_n, \dots, x_m form a polycyclic generating sequence of $X_{n,m}$. Then (ZS6) follows. From this it also follows that $X'_{n,m} \leq X_{n+1,m}$. By a symmetric argument $X'_{n,m} \leq X_{n,m-1}$ and hence (ZS5) follows. \square

Definition 4.3. Let $x \in X^*$. By (ZS6) there exist unique $n \leq m \in \mathbb{Z}$ and $e_n, \dots, e_m \in \mathbb{Z}_p$ such that $e_n \neq 0 \neq e_m$ and $x = x_n^{e_n} \cdots x_m^{e_m}$. This is the *normal form* of x , and we set

$$\mathbf{n}(x) := n, \quad \mathbf{m}(x) := m.$$

The *width* of $x \in X^*$ is $\mathbf{w}(x) := m - n + 1$. Additionally we set $\mathbf{w}(1) := 0$, $\mathbf{n}(1) := \infty$ and $\mathbf{m}(1) := -\infty$.

Finally we point out some useful direct consequences of (ZS5) and (ZS6), which we use extensively in the sequel.

Lemma 4.4. *Let $x, y \in X^*$.*

- (i) *Let $k \in \mathbb{Z}$ such that $\mathbf{n}(x) \neq k$. Then $\mathbf{n}(x_k x) = \min(k, \mathbf{n}(x))$.*
- (ii) *If $\mathbf{n}(x) = \mathbf{n}(y)$, then there is $\lambda \in \mathbb{Z}_p^*$ such that $\mathbf{n}(x) < \mathbf{n}(y^\lambda x)$ and $\mathbf{w}(y^\lambda x) < \max(\mathbf{w}(x), \mathbf{w}(y))$.*

- (iii) If $\mathbf{m}(x) = \mathbf{m}(y)$, then there is $\lambda \in \mathbb{Z}_p^*$ such that $\mathbf{m}(y^\lambda x) < \mathbf{m}(x)$ and $\mathbf{w}(y^\lambda x) < \max(\mathbf{w}(x), \mathbf{w}(y))$.
- (iv) $\mathbf{w}(x^p) < \mathbf{w}(x)$.

5 Abelian \mathbb{Z} -systems

In this section we establish a criterion for proving that a \mathbb{Z} -system is abelian, stated as Proposition 5.3.

Definition 5.1. The *lower cutoff* of Θ is defined as

$$\ell(\Theta) := \begin{cases} \infty & \text{if } X \text{ is abelian,} \\ \min\{|m - n| \mid [x_n, x_m] \neq 1\} & \text{if } X \text{ is non-abelian.} \end{cases}$$

Recall that by (ZS3) there is an automorphism t of X mapping x_n onto x_{n+2} for all $n \in \mathbb{Z}$.

Lemma 5.2. *Let X be non-abelian and let $n := \ell(\Theta)$ be the lower cutoff of Θ . If $[x_0, x_n] \neq 1$, then $[x_1, x_{n+1}] = 1$; if $[x_1, x_{n+1}] \neq 1$, then $[x_0, x_n] = 1$.*

Proof. Suppose $w := [x_0, x_n] \neq 1$. As n is the lower cutoff of Θ , the subgroup $X_{-(n-1), n-1}$ centralizes x_0 . Similarly $X_{1, 2n-1}$ centralizes x_n . Thus, for $0 \leq j < n$,

$$[x_0, x_{n+j}] \in X_{1, n+j-1} \leq X_{1, 2n-1}, \quad \text{implying} \quad [[x_0, x_{n+j}], x_n] = 1.$$

Since $j < n$ we have also $[x_n, x_{n+j}] = 1$, hence $[[x_n, x_{n+j}], x_0] = 1$. Then the Three Subgroup Lemma (see e.g. [Rob96, 5.1.10]) implies $[w, x_{n+j}] = [[x_0, x_n], x_{n+j}] = 1$.

Let $i := \mathbf{n}(w)$. As $w \in X_{1, n-1}$ it follows that $1 \leq i < n$ and hence $[w, x_{n+i}] = 1$. But w can be written as $w = x_i^{e_i} \dots x_{n-1}^{e_{n-1}}$ with $e_i, \dots, e_{n-1} \in \mathbb{Z}_p$. Since the lower cutoff is n , we have $[x_j, x_{n+1}] = 1$ for $i+1 \leq j \leq n-1$. Thus also $[x_i, x_{n+i}] = 1$.

As $[x_{2k}, x_{2k+n}] = t^k([x_0, x_n]) = t^k(w) \neq 1$ for all $k \in \mathbb{Z}$, it follows that i must be odd. So there is $m \in \mathbb{Z}$ with $i = 2m + 1$, therefore $[x_1, x_{n+1}] = t^{-m}([x_i, x_{n+i}]) = t^{-m}(1) = 1$. This proves the first assertion, the second follows by a symmetric argument. \square

Proposition 5.3. *The following are equivalent:*

- (i) *The group X is abelian.*
- (ii) *The group X is elementary abelian (i.e. abelian and of exponent p).*
- (iii) *The mapping $x_k \mapsto x_{k+1}$ extends to an automorphism of X .*

Proof. By (ZS2), the generators x_n have order p . Thus if X is abelian, then X has exponent p . Thus (i) implies (ii). The converse implication is trivial. Also that (ii) implies (iii) now is readily verified.

Assume that X is not abelian and let $n := \ell(\Theta)$. By Lemma 5.2, $[x_0, x_n] \neq 1$ implies $[x_1, x_{n+1}] = 1$ and $[x_1, x_{n+1}] \neq 1$ implies $[x_0, x_n] = 1$. Thus, the mapping $x_k \mapsto x_{k+1}$ does not extend to an automorphism of X . \square

6 Shift-invariant subgroups

In this section we study subgroups of X which are invariant under the shift map t . We prove that such subgroups are close to forming \mathbb{Z} -systems again. Moreover, those of infinite index are necessarily abelian.

Definition 6.1. A subgroup $Y \leq X$ is called *shift-invariant* if $t(Y) = Y$. We set

$$Y_{\text{even}} := \{y \in Y^* \mid \mathbf{n}(y) \in 2\mathbb{Z}\},$$

$$Y_{\text{odd}} := \{y \in Y^* \mid \mathbf{n}(y) \in 1 + 2\mathbb{Z}\}.$$

For $n \leq m \in \mathbb{Z} \cup \{\pm\infty\}$, set $Y_{n,m} := Y \cap X_{n,m}$.

Remark 6.2. By shift-invariance of Y , we have $t(Y_{n,m}) = Y_{n+2,m+2}$.

Lemma 6.3. *Let $Y \leq X$ be shift-invariant. Then the following are equivalent:*

- (i) *The index of Y in X is finite.*
- (ii) *Both Y_{even} and Y_{odd} are non-empty.*

Proof. Let Y be of finite index in X . Suppose, by contradiction, that $\mathbf{n}(y)$ is even for all $y \in Y^*$. Then $\mathbf{n}(Y^*) = 2\mathbb{Z}$ because Y is shift-invariant. In view of Lemma 4.4(i), for each odd integer m we have

$$\mathbf{n}(x_m Y) = \{m\} \cup \{2k \in 2\mathbb{Z} \mid 2k < m\}.$$

Hence for any two odd integers $m \neq m'$ we have $x_m Y \neq x_{m'} Y$. Thus we get infinitely many cosets of Y , which is a contradiction.

Similarly the assumption that $\mathbf{n}(y)$ is odd for all $y \in Y^*$ leads to a contradiction and hence (i) implies (ii).

For the converse, let a (resp. b) be of minimal width in Y_{even} (resp. Y_{odd}). Since Y is shift-invariant, we may assume that $\mathbf{n}(a) = 0$ and $\mathbf{n}(b) = 1$.

We claim that $\mathbf{m}(a)$ and $\mathbf{m}(b)$ have different parity. Suppose that this is not the case. Then there exists an element $k \in \mathbb{Z}$ such that $\mathbf{m}(t^k(a)) = \mathbf{m}(b)$. Using Lemma 4.4(iii) it follows that there is $\lambda \in \mathbb{Z}_p^*$ such that $y := b^\lambda t^k(a)$ satisfies either $y \in Y_{\text{even}}$ and

$\mathbf{w}(y) < \mathbf{w}(a)$, or $y \in Y_{\text{odd}}$ and $\mathbf{w}(y) < \mathbf{w}(b)$. Either case contradicts the minimality of a resp. b .

Let $m := \max\{\mathbf{w}(a), \mathbf{w}(b)\}$. Since $\mathbf{n}(a) = 0$ and $\mathbf{n}(b) = 1$, by using Lemma 4.4(ii) and induction, it follows that $X_{-\infty, m}Y \subseteq X_{0, m}Y$. As $\mathbf{m}(a)$ and $\mathbf{m}(b)$ have different parity, one also sees that $X_{0, \infty}Y \subseteq X_{0, m}Y$. As $X = X_{-\infty, m}X_{0, \infty}$ it follows that $X = X_{0, m}Y$. Thus $|X : Y| \leq |X_{0, m}| = p^{m+1}$. \square

Proposition 6.4. *Let $1 \neq Y \leq X$ be shift-invariant with $|X : Y| = \infty$, let $u \in Y^*$ be of minimal width in Y and $y_n := t^n(u)$ for $n \in \mathbb{Z}$. Then the following hold.*

- (i) $Y = \langle y_n \mid n \in \mathbb{Z} \rangle$.
- (ii) $(Y, (y_n)_{n \in \mathbb{Z}})$ is a \mathbb{Z} -system.
- (iii) Y is elementary abelian of exponent p .

Proof. (i) By shift-invariance of Y we have $y_n \in Y$, thus $U := \langle y_n \mid n \in \mathbb{Z} \rangle \leq Y$. For $y \in Y$ we will show by induction on $\mathbf{w}(y)$ that $y \in U$, and hence $Y = U$. If $\mathbf{w}(y) = 0$ then $y = 1 \in U$. So suppose $\mathbf{w}(y) > 0$. Now $|X : Y| = \infty$, therefore $\mathbf{n}(y)$ and $\mathbf{n}(u)$ have the same parity by Lemma 6.3. Hence there is $k \in \mathbb{Z}$ such that

$$\mathbf{n}(y) = \mathbf{n}(u) + 2k = \mathbf{n}(t^k(u)) = \mathbf{n}(y_k).$$

Moreover, $\mathbf{w}(y) \geq \mathbf{w}(y_k) = \mathbf{w}(u)$. Thus by Lemma 4.4(ii) there is $\lambda \in \mathbb{Z}_p^*$ such that $\mathbf{w}(y_k^\lambda y) < \mathbf{w}(y)$. Hence by the induction hypothesis $y_k^\lambda y \in U$. Since also $y_k \in U$ we get $y \in U$.

- (ii) (ZS1) follows from Assertion (i). (ZS3) follows from the fact that $t(Y) = Y$ and $t(y_n) = y_{n+1}$ for all $n \in \mathbb{Z}$, hence $s := t^2$ is a shift automorphism for $(Y, (y_n)_{n \in \mathbb{Z}})$. It remains to verify (ZS2). Without loss of generality, assume $\mathbf{n}(u) \in \{0, 1\}$ and thus $\mathbf{n}(y_n) \in \{2n, 2n+1\}$ for $n \in \mathbb{Z}$.

For $n \leq m \in \mathbb{Z}$ let $U_{n, m} := \langle y_n, \dots, y_m \rangle \leq X_{2n, \infty}$. As $\mathbf{n}(y_n) \in \{2n, 2n+1\}$, we have $y_n \notin X_{2n+2, \infty}$, hence $y_n \notin U_{n+1, m} \leq X_{2n+2, \infty}$. Lemma 4.4(iv) implies that $\mathbf{w}(u^p) < \mathbf{w}(u)$. Since u was of minimal width, we conclude $u^p = 1$. Thus y_n has order p . Since p is prime, we get $\langle y_n \rangle \cap U_{n+1, m} = 1$.

Now we claim that $U_{n, m} = \langle y_n \rangle U_{n+1, m}$. To see this, pick $y \in U_{n, m}$. If $\mathbf{n}(y) > \mathbf{n}(y_n)$ then $y \in U_{n+1, m}$. Otherwise $\mathbf{n}(y) = \mathbf{n}(y_n)$, and then Lemma 4.4(ii) implies that there is $\lambda \in \mathbb{Z}_p^*$ such that $\mathbf{n}(y_n^\lambda y) > \mathbf{n}(y_n)$, hence $y_n^\lambda y \in U_{n+1, m}$. The claim follows.

But $\langle y_n \rangle \cap U_{n+1, m} = 1$ and $U_{n, m} = \langle y_n \rangle U_{n+1, m}$ imply $|U_{n, m}| = p \cdot |U_{n+1, m}|$. By induction it follows that $|U_{n, m}| = p^{m-n+1}$. Thus (ZS2) holds.

- (iii) By (ii), $(Y, (y_n)_{n \in \mathbb{Z}})$ is a \mathbb{Z} -system. The shift map t of $(X, (x_n)_{n \in \mathbb{Z}})$ leaves Y invariant and thus restricts to an automorphism of Y which extends the mapping $y_k \mapsto y_{k+1}$. The claim thus follows from Proposition 5.3. \square

Lemma 6.5. *Let $Y \leq X$ be shift-invariant with $Y_{\text{even}} \neq \emptyset \neq Y_{\text{odd}}$. Let a (resp. b) be of minimal width in Y_{even} (resp. Y_{odd}) such that $\mathbf{n}(a) = 0$ and $\mathbf{n}(b) = 1$. For $n \in \mathbb{Z}$ let $y_{2n} := t^n(a)$ and $y_{2n+1} := t^n(b)$. Then the following hold:*

(i) $Y = \langle y_n \mid n \in \mathbb{Z} \rangle$.

(ii) If $\mathbf{w}(a) = \mathbf{w}(b)$, then $(Y, (y_n)_{n \in \mathbb{Z}})$ is a \mathbb{Z} -system.

Proof. (i) By shift-invariance of Y we have $y_n \in Y$, thus $U := \langle y_n \mid n \in \mathbb{Z} \rangle \leq Y$. For $y \in Y$ we will show by induction on $\mathbf{w}(y)$ that $y \in U$, and hence $Y = U$. If $\mathbf{w}(y) = 0$ then $y = 1 \in U$. So suppose $\mathbf{w}(y) > 0$ and let $n := \mathbf{n}(y)$. Then $\mathbf{w}(y) \geq \mathbf{w}(y_n)$. Since $\mathbf{n}(y_n) = n = \mathbf{n}(y)$, by Lemma 4.4(ii) there is $\lambda \in \mathbb{Z}_p^*$ such that $\mathbf{w}(y_n^\lambda y) < \mathbf{w}(y)$. Hence by the induction hypothesis $y_n^\lambda y \in U$. Since also $y_n \in U$ we get $y \in U$.

(ii) This follows by a similar argument as in the proof of Assertion (ii) in Proposition 6.4. (Note that we do not make use of this observation in this paper.) \square

Combining the previous statements yields the following:

Lemma 6.6. *Let Y be a shift-invariant subgroup of X . Then there are elements $a, b \in Y$ such that $Y = \langle t^k(a), t^k(b) \mid k \in \mathbb{Z} \rangle$.*

Proof. If Y has finite index in X , this follows from Lemmas 6.3 and 6.5. If Y is trivial, we can choose $a = b = 1$. Finally, if Y is non-trivial but has infinite index, this follows from Proposition 6.4 \square

Remark 6.7. We can make the choice of generators a, b unique by requiring that each should either be trivial; or else start at index 0 or 1, be of minimal width amongst all such elements, and have “lead exponent” equal to 1.

The resulting generating system is close to being a \mathbb{Z} -system again. However, the generators are not necessarily independent anymore; in particular, it can happen that $a^p = b$.

Lemma 6.8. *Let Y be a shift-invariant subgroup of X . Then for every $n \in \mathbb{Z}$, there is $m \in \mathbb{Z}$ such that $Y = Y_{-\infty, m} Y_{n, \infty}$.*

Proof. Pick $a, b \in Y$ as in Lemma 6.6. Since Y is generated by all shifts of a and b , it suffices to choose m large enough such that $Y_{-\infty, m}$ contains all the shifts of a and b which are not in $Y_{n, \infty}$. For example, choose $m := \max\{n + \mathbf{w}(a), n + \mathbf{w}(b)\}$. \square

7 One-sided normal subgroups

Throughout this section, let Y be a shift-invariant subgroup of X .

Notation 7.1. Let G be a group. The *normal closure* of $U \subseteq G$ is $\langle U \rangle^G := \langle U^G \rangle = \langle g^{-1}Ug \mid g \in G \rangle$.

Remark 7.2. Recall that a group G is *locally nilpotent* if every finitely generated subgroup of G is nilpotent. Now every finitely generated subgroup H of X is contained in some $X_{n,m}$ with $n \leq m \in \mathbb{Z}$, which is a finite p -group by (ZS2). Hence H is a finite p -group, and X is locally nilpotent.

Lemma 7.3. Let K be nilpotent and $A \leq K$ with $A \leq [A, K]$. Then $A = 1$.

Proof. K is nilpotent, hence its lower central series $K \supseteq [K, K] \supseteq [K, K, K] \supseteq \dots$ vanishes after finitely many steps. Since $A \leq K$, also $[A, K, \dots, K]$ eventually vanishes. From $A \leq [A, K]$ we deduce, by forming the commutator with K , that

$$A \leq [A, K] \leq [A, K, K] \leq \dots \leq 1. \quad \square$$

Lemma 7.4. Let G be a locally nilpotent group and let $A \leq G$ be finitely generated. Then $A \leq \langle [A, G] \rangle^G$ if and only if $|A| = 1$.

Proof. The implication starting with $|A| = 1$ is obvious. So suppose $A \leq \langle [A, G] \rangle^G$ and $A = \langle a_1, \dots, a_n \rangle$. Then for $1 \leq i \leq n$, there exist $\ell_i \in \mathbb{N}$ and elements $a_{ij} \in A$, $g_{ij}, h_{ij} \in G$ such that

$$a_i = [a_{i1}, g_{i1}]^{h_{i1}} \dots [a_{i\ell_i}, g_{i\ell_i}]^{h_{i\ell_i}}. \quad (1)$$

We now define the finitely generated subgroup

$$H := \langle h_{ij}, g_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq \ell_i \rangle.$$

Moreover, we set $K := \langle A, H \rangle$, and observe that

$$A^H := \langle A \rangle^H = \langle a^h \mid a \in A, h \in H \rangle \leq K.$$

Since A and H are finitely generated, so is K , hence K is nilpotent. From Equation (1) we then conclude $A \leq [A^H, H]$ hence $A^H \leq [A^H, H] \leq [A^H, K]$. Applying Lemma 7.3, we conclude that $A^H = 1$. Hence $A = 1$. \square

Lemma 7.5. Let $n \in \mathbb{Z}$. Then there is $y_{n-1} \in Y_{n-1, \infty}$ such that

$$Y_{n-1, \infty} = \langle y_{n-1}, Y_{n, \infty} \rangle.$$

Moreover, for any $N \geq \mathbf{w}(y_{n-1}) - 2$, we have

$$Y_{n-1, n+N} = \langle y_{n-1}, Y_{n, n+N} \rangle.$$

Proof. If $Y_{n-1,\infty} = Y_{n,\infty}$ set $y_{n-1} := 1$ and the first assertion clearly holds. Otherwise there exists $y_{n-1} \in Y_{n-1,\infty}$ with $\mathbf{n}(y_{n-1}) = n - 1$. Let $y \in Y_{n-1,\infty}$. If $\mathbf{n}(y) \geq n$, then $y \in Y_{n,\infty}$. Otherwise, if $\mathbf{n}(y) = n - 1$, then by Lemma 4.4(ii) there is $\lambda \in \mathbb{Z}_p^*$ such that $\mathbf{n}(y_{n-1}^\lambda y) \geq n$, hence $y \in \langle y_{n-1}, Y_{n,\infty} \rangle$. Thus $Y_{n-1,\infty} \leq \langle y_{n-1}, Y_{n,\infty} \rangle$. The reverse inclusion is obvious.

The second assertion follows analogously, after observing that $y_{n-1} \in Y_{n-1,n+N}$. Indeed, $\mathbf{n}(y_{n-1}) = n - 1$ and $\mathbf{m}(y_{n-1}) = \mathbf{w}(y_{n-1}) + \mathbf{n}(y_{n-1}) - 1 = n + (\mathbf{w}(y_{n-1}) - 2)$. \square

Lemma 7.6. *Let $n \in \mathbb{Z}$.*

- (i) *If $Y_{n-1,\infty} \leq \langle Y_{n,\infty} \rangle^X$, then there is $M \in \mathbb{N}$ such that $Y_{n-1,n+N} \leq \langle Y_{n,n+N} \rangle^X$ for all $N \geq M$.*
- (ii) *If $Y_{-\infty,n+1} \leq \langle Y_{-\infty,n} \rangle^X$, then there is $M \in \mathbb{N}$ such that $Y_{n-N,n+1} \leq \langle Y_{n-N,n} \rangle^X$ for all $N \geq M$.*

Proof. We prove the first case, the second follows by a symmetric argument. Suppose $Y_{n-1,\infty} \leq \langle Y_{n,\infty} \rangle^X$. If $Y_{n-1,\infty} = Y_{n,\infty}$ we are done, as then $Y_{n-1,n+N} = Y_{n,n+N}$ for all $N \in \mathbb{N}$. Otherwise, let $y_{n-1} \neq 1$ be as in Lemma 7.5. Since $y_{n-1} \in Y_{n-1,\infty} \leq \langle Y_{n,\infty} \rangle^X$, there are $\ell \in \mathbb{N}$, $a_1, \dots, a_\ell \in Y_{n,\infty}^*$ and $g_1, \dots, g_\ell \in X$ such that $y_{n-1} = a_1^{g_1} \cdots a_\ell^{g_\ell}$. Let $M := \max\{\mathbf{m}(a_1), \dots, \mathbf{m}(a_\ell), \mathbf{m}(y_{n-1})\} - n$. Then $a_1, \dots, a_\ell \in Y_{n,n+M}$, hence for $N \geq M$ we have

$$y_{n-1} \in \langle Y_{n,n+M} \rangle^X \leq \langle Y_{n,n+N} \rangle^X.$$

Moreover, by definition

$$M \geq \mathbf{m}(y_{n-1}) - n = \mathbf{m}(y_{n-1}) - \mathbf{n}(y_{n-1}) - 1 = \mathbf{w}(y_{n-1}) - 2.$$

Thus for $N \geq M$, Lemma 7.5 yields

$$Y_{n-1,n+N} = \langle y_{n-1}, Y_{n,n+N} \rangle \leq \langle Y_{n,n+N} \rangle^X. \quad \square$$

Lemma 7.7.

- (i) *If $Y_{n-1,\infty} \leq \langle Y_{n,\infty} \rangle^X$ for all $n \in \mathbb{Z}$, then there is $M \in \mathbb{N}$ with $Y_{-\infty,M} \leq \langle Y_{0,M} \rangle^X$.*
- (ii) *If $Y_{-\infty,n+1} \leq \langle Y_{-\infty,n} \rangle^X$ for all $n \in \mathbb{Z}$, then there is $M \in \mathbb{N}$ with $Y_{0,\infty} \leq \langle Y_{0,M} \rangle^X$.*

Proof. We prove the first case, the second follows by a symmetric argument. The hypothesis implies for all $n \in \mathbb{N}$ that

$$Y_{n-2,\infty} \leq \langle Y_{n-1,\infty} \rangle^X \quad \text{and} \quad Y_{n-1,\infty} \leq \langle Y_{n,\infty} \rangle^X, \quad \text{hence} \quad Y_{n-2,\infty} \leq \langle Y_{n,\infty} \rangle^X.$$

Thus for all $n, k \in \mathbb{N}$ we have

$$Y_{n-k,\infty} \leq \langle Y_{n,\infty} \rangle^X.$$

But this implies $Y \leq \langle Y_{n,\infty} \rangle^X$. By Lemma 6.6, there are elements $a, b \in Y$ such that $Y = \langle t^k(a), t^k(b) \mid k \in \mathbb{Z} \rangle$. As Y is shift-invariant, we may assume $\mathbf{n}(a) = -2$ or $a = 1$, and $\mathbf{n}(b) = -1$ or $b = 1$. Then

$$a, b \in Y_{-2,\infty} \leq \langle Y_{0,\infty} \rangle^X.$$

By applying Lemma 7.6 twice we deduce the existence of some value $M \in \mathbb{N}$ such that $a, b \in \langle Y_{0,M} \rangle^X$. But then for all $N \geq M$

$$Y_{-2,N} = \langle a, b, Y_{0,N} \rangle \leq \langle Y_{0,N} \rangle^X. \quad (2)$$

By shift-invariance, we can now conclude that

$$Y_{-4,M} = t^{-1}(Y_{-2,M+2}) \stackrel{(2)}{\leq} t^{-1}(\langle Y_{0,M+2} \rangle^X) = \langle Y_{-2,M} \rangle^X \stackrel{(2)}{\leq} \langle Y_{0,M} \rangle^X.$$

By induction it follows that $Y_{-\infty,M} \leq \langle Y_{0,M} \rangle^X$. \square

Lemma 7.8. *If for all $n \in \mathbb{Z}$ we have*

$$Y_{n-1,\infty} \leq \langle Y_{n,\infty} \rangle^X \quad \text{and} \quad Y_{-\infty,n+1} \leq \langle Y_{-\infty,n} \rangle^X$$

then there exists $M \in \mathbb{N}$ such that $Y \leq \langle Y_{0,M} \rangle^X$.

Proof. This is an immediate consequence of Lemma 7.7. \square

Proposition 7.9. *Suppose $[X, Y] = Y$. Then either $Y = 1$, or there exists $n \in \mathbb{Z}$ such that at least one of the following holds:*

- (i) $Y_{n-1,\infty} \not\leq \langle Y_{n,\infty} \rangle^X$.
- (ii) $Y_{-\infty,n+1} \not\leq \langle Y_{-\infty,n} \rangle^X$.

Proof. Suppose the claim is false. Then by Lemma 7.8 there is $M \in \mathbb{N}$ such that $Y \leq \langle Y_{0,M} \rangle^X$. The group $Y_{0,M}$ is finite, so we can pick a finite generating set $Y_{0,M} = \langle z_1, \dots, z_k \rangle$. Then for $1 \leq i \leq k$, since $z_i \in Y = [X, Y]$, there are $\ell_i \in \mathbb{N}$ and $y_{ij} \in Y$, $g_{ij} \in X$ for $1 \leq j \leq \ell_i$ such that

$$z_i = [g_{i1}, y_{i1}] \cdots [g_{i\ell_i}, y_{i\ell_i}].$$

Let

$$n := \min\{\mathbf{n}(y_{ij}) \mid 1 \leq i \leq k, 1 \leq j \leq \ell_i\},$$

$$m := \max\{\mathbf{m}(y_{ij}) \mid 1 \leq i \leq k, 1 \leq j \leq \ell_i\}.$$

Then $z_i \in [X, Y_{n,m}]$, hence $Y_{0,M} \subseteq [X, Y_{n,m}]$. But then

$$Y_{n,m} \leq Y \leq \langle Y_{0,M} \rangle^X \leq \langle [X, Y_{n,m}] \rangle^X. \quad (3)$$

Since X is locally nilpotent, Lemma 7.4 implies $Y_{n,m} = 1$. Inserting this into Equation (3) yields $Y = 1$. \square

Lemma 7.10. *Suppose $Y \trianglelefteq X$ and $|X : Y| = \infty$. Then the following hold.*

- (i) *If $Y_{n,\infty} \not\leq \langle Y_{n+1,\infty} \rangle^X$ for some $n \in \mathbb{Z}$, then $Y_{n,\infty} \trianglelefteq X$.*
- (ii) *If $Y_{-\infty,n} \not\leq \langle Y_{-\infty,n-1} \rangle^X$ for some $n \in \mathbb{Z}$, then $Y_{-\infty,n} \trianglelefteq X$.*

Proof. We prove the first case, the second follows by a symmetric argument. By Proposition 6.4 there is $y \in Y$ such that $(Y, (t^k(y))_{k \in \mathbb{Z}})$ is a \mathbb{Z} -system. Suppose now that there is $n \in \mathbb{Z}$ such that $Y_{n,\infty} \not\leq \langle Y_{n+1,\infty} \rangle^X$. Then as Y is shift-invariant, we may assume that $\mathbf{n}(y) = n$, and so $y \in Y_{n,\infty}$ but $y \notin \langle Y_{n+1,\infty} \rangle^X$.

Suppose now that there is $m < n$ with $[x_m, y] \neq 1$. Then there are integers $i_1 < \dots < i_s < 0$ and exponents $e_1, \dots, e_s \in \mathbb{Z}_p^*$, such that $[x_m, y] = t^{i_1}(y)^{e_1} \dots t^{i_s}(y)^{e_s}$. Applying t^{-i_1} we get

$$[x_{m-2i_1}, t^{-i_1}(y)] = y^{e_1} \cdot t^{i_2-i_1}(y)^{e_2} \dots t^{i_s-i_1}(y)^{e_s}$$

and, since $-i_1, i_2 - i_1, \dots, i_s - i_1$ all are positive, we conclude

$$y^{e_1} = [x_{m-2i_1}, t^{-i_1}(y)] \cdot (t^{i_2-i_1}(y)^{e_2} \dots t^{i_s-i_1}(y)^{e_s})^{-1} \in \langle Y_{n+1,\infty} \rangle^X.$$

But $\mathbf{n}(y^{e_1}) = \mathbf{n}(y) = n$, thus $Y_{n,\infty} = \langle y^{e_1}, Y_{n+1,\infty} \rangle \leq \langle Y_{n+1,\infty} \rangle^X$, contradicting the hypothesis. Therefore $[x_m, y] = 1$ for all $m < n$. Since $Y_{n,\infty} = \langle t^i(y) \mid i \in \mathbb{N} \rangle$, we get $X_{-\infty,n-1} \leq C_X(Y_{n,\infty})$ and so, using that $Y \trianglelefteq X$,

$$[X, Y_{n,\infty}] = [X_{n,\infty}, Y_{n,\infty}] \leq [X_{n,\infty}, Y] \cap X_{n,\infty} \leq Y \cap X_{n,\infty} = Y_{n,\infty}. \quad \square$$

Thus we obtain the main result of this section:

Proposition 7.11. *Let $(X, (x_n)_{n \in \mathbb{Z}})$ be a \mathbb{Z} -system of prime order p . Suppose Y is a shift-invariant subgroup of X , with $|X : Y| = \infty$ and $[X, Y] = Y$. Then there is $n \in \mathbb{Z}$ such that $Y_{n,\infty} \trianglelefteq X$ or $Y_{-\infty,n} \trianglelefteq X$, where $Y_{n,\infty} := Y \cap X_{n,\infty}$ and $Y_{-\infty,n} := Y \cap X_{-\infty,n}$.*

Proof. This follows by first applying Proposition 7.9, then Lemma 7.10. \square

8 Infinite abelianization

Notation 8.1. Let G be a group. Then let $G^{(0)} := G$, let $G' := [G, G]$ be the *derived subgroup* and for $k \in \mathbb{N}$ let $G^{(k+1)} := [G^{(k)}, G^{(k)}]$.

Lemma 8.2. *Let $1 \neq Y \trianglelefteq X$ be shift-invariant. Then $[Y, Y] < Y$.*

Proof. Suppose that $[Y, Y] = Y$. Then we have also $[X, Y] = Y$. Since also $Y \neq 1$, by Proposition 7.9 this implies that there exists $n \in \mathbb{Z}$ such that $Y_{n-1, \infty} \not\leq \langle Y_{n, \infty} \rangle^X$ or $Y_{-\infty, n+1} \not\leq \langle Y_{-\infty, n} \rangle^X$ holds. Suppose that $Y_{n-1, \infty} \not\leq \langle Y_{n, \infty} \rangle^X$ (the other case is dealt with by a symmetric argument).

Let $N := \langle Y_{n, \infty} \rangle^X$. Then $N \trianglelefteq X$ and by what we just said $Y_{n-1, \infty} \not\leq N$, thus $Y \neq N$. On the other hand, from $Y_{n, \infty} \leq Y \trianglelefteq X$ it follows that $N \trianglelefteq Y$.

By Lemma 6.8 there is $m \in \mathbb{Z}$ such that

$$Y \stackrel{6.8}{=} Y_{-\infty, m} Y_{n, \infty} \leq Y_{-\infty, m} N \leq Y,$$

hence $Y = Y_{-\infty, m} N$. Choose $m \in \mathbb{N}$ minimal with this property. Then $Y'_{-\infty, m} \leq Y_{-\infty, m-1}$ by (ZS5) and so

$$[Y, Y] = Y' \leq Y'_{-\infty, m} N \leq Y_{-\infty, m-1} N < Y,$$

a contradiction. \square

Corollary 8.3. *For $k \in \mathbb{N}$, we have $|X : X^{(k)}| \geq p^k$.*

Proof. The claim follows by induction on k , and the following observations: $X^{(k)}$ is a characteristic subgroup of X , hence shift-invariant and normal. Thus if $X^{(k)} \neq 1$, then $X^{(k+1)} < X^{(k)}$ by Lemma 8.2. And if $X^{(k)} = 1$, then $|X : X^{(k)}| = |X : 1| = |X| = \infty$. \square

Lemma 8.4 ([MKS66, Lemma 5.9]). *Let G be a nilpotent group. If $z_1, \dots, z_\ell \in G$ satisfy $G/G' = \langle z_1 G', \dots, z_\ell G' \rangle$, then $G = \langle z_1, \dots, z_\ell \rangle$.*

Lemma 8.5. *There is $k \in \mathbb{N}$ such that $|X : X^{(k)}| = \infty$.*

Proof. Suppose $|X : X^{(k)}| < \infty$ for all $k \in \mathbb{N}$. Choose $z_1, \dots, z_\ell \in X$ such that $X/X' = \langle z_1 X', \dots, z_\ell X' \rangle$. For $k \in \mathbb{N}$, the groups $G_k := X/X^{(k)}$ are finite p -groups and hence nilpotent. Next observe that

$$G_k/G'_k \cong X/X' = \langle z_1 X', \dots, z_\ell X' \rangle$$

implies that

$$G_k/G'_k = \langle \hat{z}_1 G'_k, \dots, \hat{z}_\ell G'_k \rangle,$$

where $\hat{z}_1 := z_1 X^{(k)}, \dots, \hat{z}_\ell := z_\ell X^{(k)}$. Therefore, by Lemma 8.4 we conclude

$$G_k = \langle z_1 X^{(k)}, \dots, z_\ell X^{(k)} \rangle.$$

Now let $Z := \langle z_1, \dots, z_\ell \rangle \leq X$. Since X is locally finite, $|Z| < \infty$. It follows that $X = ZX^{(k)}$ for all $k \in \mathbb{N}$, hence $|X : X^{(k)}| \leq |Z|$. But this is a contradiction, as $|X : X^{(k)}|$ becomes arbitrarily large by Corollary 8.3. \square

Lemma 8.6 ([Rob96, 5.2.6]). *A nilpotent group G with $|G : G'| < \infty$ is finite.*

Lemma 8.7. *Let G be a p -group, $N \trianglelefteq G$ nilpotent of finite exponent and $|G : N| < \infty$. Then G is nilpotent of finite exponent.*

Proof. We will assume $|G : N| = p$, the general case follows by induction on $|G : N|$. Let

$$Z_0 := 1 \leq Z_1 := Z(N) \leq Z_2 \leq \cdots \leq Z_n = N$$

be the upper central series of N . Then for all i , the Z_i are characteristic in N and hence normal in G . Since N has finite exponent, we can refine this series to a series

$$W_0 := 1 \leq W_1 \leq \cdots \leq W_m = N$$

such that W_i is normal in G and $M_i := W_i/W_{i-1}$ has exponent p for all $i > 0$. In fact, since we refined a central series, the M_i are elementary abelian p -groups, in other words, vector spaces over a finite field of order p .

Let $x \in G \setminus N$. Since N acts trivially on M_i , and since $|G : N| = p$, it follows for all $i > 0$ that x induces an automorphism x_i of order at most p on the vector space M_i . Since $x_i^p = 1$, the linear map x_i has a minimal polynomial dividing $t^p - 1 = (t - 1)^p$.

But then $[v, \underbrace{x_i, \dots, x_i}_p] = 1$ for all $v \in M_i$. Hence we can refine the series in such a way

that G acts trivially on each factor. Therefore G is nilpotent, and since N and G/N have finite exponent, the exponent of G is also finite. \square

Remark 8.8. Note that the condition that the exponent of N is finite is essential. For example, let G be the injective limit of dihedral groups $(D_{2^n})_{n \geq 1}$, that is

$$G = \langle s, r_1, r_2, r_3, \dots \mid s^2 = 1 = r_1^2, r_{n+1}^2 = r_n, r_n s = s r_n^{-1} \text{ for } n \geq 1 \rangle.$$

Let N the normal subgroup generated by the rotations r_n . Then N is an abelian 2-group and G/N has order 2, but $[G, N] = N$.

Theorem 8.9. *Let $(X, (x_n)_{n \in \mathbb{Z}})$ be a \mathbb{Z} -system of prime order p . Then X has infinite abelianization X/X' .*

Proof. For $k \in \mathbb{N}$, let $G_k := X/X^{(k)}$ and $H_k := X^{(k)}/X^{(k+1)}$. Since G_0 is trivial, Lemma 8.5 implies that there is $k \in \mathbb{N}$ such that $|G_k| < \infty$ and $|G_{k+1}| = \infty$. We have $X^{(k+1)} \leq X^{(k)} \leq X$ and therefore

$$|G_{k+1}| = |X : X^{(k+1)}| = |X : X^{(k)}| \cdot |X^{(k)} : X^{(k+1)}| = |G_k| \cdot |H_k|.$$

Thus $|H_k| = \infty$.

Since $X^{(k)}$ is shift-invariant, by Lemma 6.6 it is generated by the shifts of two elements $a, b \in X^{(k)}$, that is

$$H_k = \langle t^m(a)X^{(k+1)}, t^m(b)X^{(k+1)} \mid m \in \mathbb{Z} \rangle.$$

Since H_k is an abelian p -group, there is $n \in \mathbb{N}$ such that these generators all have orders dividing p^n . Thus H_k has finite exponent and as $|G_{k+1} : H_k| = |G_k| < \infty$, the group G_{k+1} is nilpotent by Lemma 8.7. But G_{k+1} is infinite, so $G_{k+1}/G'_{k+1} \cong X/X'$ must also be infinite by Lemma 8.6. \square

9 Nilpotency class 2

Lemma 9.1. *Let $Y \trianglelefteq X$, $y, y' \in Y$ and $x \in X$. Then $[yy', x] \in [y, x][y', x][Y, X, X]$.*

Proof. We have $[Y, X, X] \trianglelefteq X$, hence

$$[yy', x] = [y, x]^{y'}[y', x] = [y, x] \underbrace{[y, x]^{-1}[y, x]^{y'}}_{=[y, x], y'}[y', x] \in [y, x][y', x][Y, X, X]. \quad \square$$

Lemma 9.2. *Let $Y \trianglelefteq X$ be shift-invariant, and suppose $|X : Y| = \infty$. Then $[Y, X, X] = [Y, X]$.*

Proof. For $Y = 1$ the claim is obvious, so we suppose $Y \neq 1$. Since $[Y, X, X] \leq [Y, X]$, it suffices to show the reverse inclusion.

As $|X : Y| = \infty$, by Proposition 6.4 the shifts of any element $y \in Y^*$ of minimal width in Y^* generate the group Y , which is abelian. Set $n := \mathbf{n}(y)$ and $m := \mathbf{m}(y)$. Then $Y_{n+1, m} = 1$ as y is of minimal width in Y^* . We will now show by induction on $N \geq n$ that $[y, x_N] \in [Y, X, X]$. Indeed, for $n \leq N \leq m+1$, we have $[y, x_N] \in Y_{n+1, m} = 1 \leq [Y, X, X]$.

So suppose $N > m+1$, and $[y, x_N] \neq 1$. Since $[y, x_N] \in Y_{n+1, N-1}$, applying (ZS6) to the \mathbb{Z} -system $(Y, t^k(y)_{k \in \mathbb{Z}})$ yields that there are uniquely determined values $s \in \mathbb{N}$, $i_1, \dots, i_s \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_s \in \mathbb{Z}_p^*$ such that

$$0 < 2i_1 < \dots < 2i_s \leq N - 1 - m \quad \text{and} \quad [y, x_N] = t^{i_1}(y)^{\lambda_1} \dots t^{i_s}(y)^{\lambda_s}. \quad (4)$$

If $s > 1$, then for $k = 2, \dots, s$, the preceding inequality together with $0 < i_1 < i_k$ implies

$$m + 1 \leq N - 2i_k < N + 2i_1 - 2i_k = N - 2(i_k - i_1) < N,$$

hence by the induction hypothesis and by the shift-invariance of $[Y, X, X]$ we have

$$[t^{i_k}(y), x_{N+2i_1}] = t^{i_k}([y, x_{N+2i_1-2i_k}]) \in [Y, X, X]. \quad (5)$$

Applying Lemma 9.1 repeatedly, we find

$$\begin{aligned}
[[y, x_N], x_{N+2i_1}] &\stackrel{(4)}{=} [t^{i_1}(y)^{\lambda_1} \cdots t^{i_s}(y)^{\lambda_s}, x_{N+2i_1}] \\
&\stackrel{9.1}{\in} [t^{i_1}(y)^{\lambda_1}, x_{N+2i_1}] \cdots [t^{i_s}(y)^{\lambda_s}, x_{N+2i_1}][Y, X, X] \\
&\stackrel{(5)}{=} [t^{i_1}(y)^{\lambda_1}, x_{N+2i_1}][Y, X, X] \\
&\stackrel{9.1}{=} [t^{i_1}(y), x_{N+2i_1}]^{\lambda_1}[Y, X, X] \\
&= t^{i_1}([y, x_N])^{\lambda_1}[Y, X, X] \\
&= t^{i_1}([y, x_N]^{\lambda_1})[Y, X, X].
\end{aligned}$$

Therefore $t^{i_1}([y, x_N]^{\lambda_1}) \in [Y, X, X]$. But $[Y, X, X]$ is shift-invariant, hence we also have $[y, x_N]^{\lambda_1} \in [Y, X, X]$. And Y has prime exponent p , thus also $[y, x_N] \in [Y, X, X]$. This concludes the proof of the claim that $[y, x_N] \in [Y, X, X]$ for all $N \geq n$.

A similar argument shows that $[y, x_N] \in [Y, X, X]$ also holds for all $N < n$. But $[Y, X] = \langle t^k([y, x_N]) \mid k, N \in \mathbb{Z} \rangle^X$, therefore $[Y, X] = [Y, X, X]$. \square

Remark 9.3. Suppose that G and V are groups and that G acts on V from the right by automorphisms. Then we define

$$[V, G] := \langle v^{-1} \cdot v^g \mid g \in G, v \in V \rangle.$$

This is a natural extension of the commutator group notation, e.g. for $V \trianglelefteq G$.

Lemma 9.4. *Let G and V be p -groups, with G acting on V by automorphisms. If V is finite and non-trivial, then $[V, G]$ is a proper subgroup of V .*

Proof. Let $\alpha : G \rightarrow \text{Aut}(V)$ be the action homomorphism associated to the action of G on V . Since V is finite, also $\text{Aut}(V)$ is finite, and hence $\tilde{G} := \alpha(G)$ is finite. Clearly $[V, G] = [V, \tilde{G}]$. Form the semidirect product $K := V \rtimes \tilde{G}$. Then $[V, \tilde{G}] \leq [V, K]$. Since $V \trianglelefteq K$ we have $[V, K] \leq V$. Moreover, K is a finite p -group, and thus it is nilpotent. Hence if $[V, K] = V$, then by Lemma 7.3 we get $V = 1$, a contradiction. Thus

$$[V, G] = [V, \tilde{G}] \leq [V, K] < V. \quad \square$$

Lemma 9.5. *Let $Y \trianglelefteq X$ be shift-invariant with $[X, Y] \neq 1$. Suppose there is $y \in Y^*$ such that $(Y, (t^k(y))_{k \in \mathbb{Z}})$ is a \mathbb{Z} -system. Then Y is elementary abelian, and for $m := \mathbf{m}(y)$, the group $M := Y_{-\infty, m} = Y \cap X_{-\infty, m}$ is an $\mathbb{F}_p X_{-\infty, m}$ -module, and $M_0 := [M, X_{-\infty, m}]$ is a proper, non-trivial submodule of finite index.*

Proof. The group Y is elementary abelian by Proposition 5.3, hence so is M . As $Y \trianglelefteq X$, the group M is an $\mathbb{F}_p X_{-\infty, m}$ -module. We compute

$$[Y, X] = \bigcup_{k \in \mathbb{N}} [Y_{-\infty, m+2k}, X_{-\infty, m+2k}] = \bigcup_{k \in \mathbb{N}} t^k(M_0).$$

The hypothesis states $[X, Y] \neq 1$, so we must have $M_0 \neq 1$. Moreover $y \in M$, but

$$M_0 = [Y_{-\infty, m}, X_{-\infty, m}] \leq [X_{-\infty, m}, X_{-\infty, m}] \leq X_{-\infty, m-1},$$

and $y \notin X_{-\infty, m-1}$, hence $y \notin M_0$. We conclude that $M_0 \neq M$, i.e. M_0 is a proper, non-trivial submodule.

Since $M = \langle t^{-k}(y) \mid k \in \mathbb{N} \rangle$, we may also regard M as an $\mathbb{F}_p[t^{-1}]$ -module, which is generated by $y \in M$. Hence it is a free $\mathbb{F}_p[t^{-1}]$ -module of rank 1. Now M_0 is a proper non-trivial $\mathbb{F}_p[t^{-1}]$ -submodule of M , thus M_0 must have finite index in M . \square

We are now ready to prove our main theorem.

Proof of Theorem 3.4. Set $Y := [X, X, X]$. Our goal is to prove $Y = 1$. Clearly $Y \trianglelefteq X$ and also $Y \trianglelefteq X'$ hold. By Theorem 8.9 we have $|X : X'| = \infty$. We thus may apply Lemma 9.2 for X' , which yields

$$Y \stackrel{\text{def.}}{=} [X', X] \stackrel{9.2}{=} [X', X, X] \stackrel{\text{def.}}{=} [Y, X].$$

In addition, $Y \leq X'$ and $|X : X'| = \infty$ imply $|X : Y| = \infty$. Therefore Proposition 7.11 is applicable, and proves that there is $n \in \mathbb{Z}$ such that $Y_{n, \infty} \trianglelefteq X$ or $Y_{-\infty, n} \trianglelefteq X$. We may assume (up to a relabeling of the generators of X) without loss of generality that the first case holds.

We proceed by assuming that $Y \neq 1$ and derive a contradiction. By Proposition 6.4 there is $y \in Y^*$ with $\mathbf{n}(y) = n$ and $Y_{n, \infty} = \langle t^k(y) \mid k \in \mathbb{N} \rangle$. Let

$$N := Y_{n+2, \infty}, \quad m := \mathbf{m}(y), \quad Y_0 := [Y/N, X_{-\infty, m}],$$

where we regard Y/N as an $\mathbb{F}_p X$ -module, which is feasible since $Y \trianglelefteq X$ and also

$$N = t(Y_{n, \infty}) \trianglelefteq t(X) = X.$$

We claim that Y_0 is an $\mathbb{F}_p X$ -submodule of Y/N . Indeed, we have

$$[X_{-\infty, m}, X] \leq X' \leq C_X(Y), \quad \text{implying} \quad X_{-\infty, m}^g \subseteq X_{-\infty, m} C_X(Y)$$

for all $g \in X$. Moreover, from $Y = Y^g$ and $[a, bc] = [a, c][a, b]^c$ it follows that

$$[Y, X_{-\infty, m}]^g = [Y^g, X_{-\infty, m}^g] \leq [Y, X_{-\infty, m} C_X(Y)] = [Y, X_{-\infty, m}].$$

Hence Y_0 is indeed an $\mathbb{F}_p X$ -submodule of Y/N .

By Lemma 9.5, we have $1 < |M : M_0| < \infty$ for

$$M := Y_{-\infty, m}, \quad M_0 := [M, X_{-\infty, m}].$$

Since Y/N is an $\mathbb{F}_p X$ -module, it is also an $X_{-\infty, m}$ -module. In fact Y/N and M are isomorphic as $X_{-\infty, m}$ -modules: Indeed, Y is the inner direct product of N and M , thus we get the isomorphism

$$M \rightarrow Y/N, \quad g \mapsto gN.$$

This isomorphism maps M_0 to Y_0 , and so Lemma 9.5 implies $1 < |Y/N : Y_0| < \infty$.

Therefore $A := (Y/N)/Y_0$ is a non-trivial, finite p -group on which X acts by automorphisms, and so Lemma 9.4 implies $[A, X] < A$. Yet earlier on we proved $[Y, X] = Y$, which implies

$$[A, X] = [(Y/N)/Y_0, X] = (Y/N)/Y_0 = A.$$

But this is a contradiction. Hence our initial assumption that $Y \neq 1$ was wrong, and so Y is trivial. Since by definition $Y = [X, X, X]$, this completes the claim. \square

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